Perturbative calculation of quasi-normal modes
George Siopsis

OUTLINE

• Introduction
• Schwarzschild black holes
• Kerr black holes
• 3D AdS space
• 5D AdS space
• Conclusions and Future

PAPERS

S. Musiri and G. S.,

Introduction

Quasi-normal modes (QNMs) describe small perturbations of a black hole.

- A black hole is a thermodynamical system whose (Hawking) temperature and entropy are given in terms of its global characteristics (total mass, charge and angular momentum).

QNMs obtained by solving a wave equation for small fluctuations subject to the conditions that the flux be

- ingoing at the horizon and
- outgoing at asymptotic infinity.

⇒ discrete spectrum of complex frequencies.

- imaginary part determines the decay time of the small fluctuations

\[ \Im \omega \equiv \frac{1}{\tau} \]
Schwarzschild black holes study QNMs in asymptotically flat space-times

- asymptotic form of QNM frequencies is related to the Barbero-Immirzi parameter of Loop Quantum Gravity.

Asymptotic form of QNMs:

\[
\frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln 3
\]

- derived numerically
  \[ Chandrasekhar and Detweiler; Leaver; Nollert; Andersson; Bachelot and Motet-Bachelot \]

- subsequently confirmed analytically
  \[ Motl and Neitzke \]

\( \Im \omega_n \) is large

\[ \Rightarrow \] numerical analysis cumbersome

\[ \Rightarrow \] easy to understand \( \because \) spacing of frequencies is \( 2\pi iT_H \)

- same as spacing of poles of a thermal Green function on the Schwarzschild black hole background.
\( \Re \omega_n \) is small

- Analytical value proposed by Hod.

Number of microstates is related to Bekenstein-Hawking entropy

\[
g_n = e^{S_{BH}} \sim k^n, \quad k = 2, 3, \ldots
\]

[Mukhanov and Bekenstein]

spacing of eigenvalues

\[
e^{\delta S_{BH}} = \frac{g_{n+1}}{g_n} \implies \delta S_{BH} = \ln k
\]

Area spectrum of black holes

\[
\delta A = 4G \ln k, \quad k = 2, 3, \ldots
\]

since

\[
S_{BH} = \frac{1}{4G} A
\]

Bohr’s correspondence principle

\[
\delta M = \hbar \Re \omega
\]

and first law of black hole mechanics

\[
\delta M = T_H \delta S_{BH}
\]
imply

\[ \delta S = \frac{\Re \omega}{T_H} = \ln 3 \]

\[ \therefore \]

\[ k = 3 \]

Intriguing value from LQG point of view:

⇒ gauge group should be \( SO(3) \) rather than \( SU(2) \)

\[ \therefore k = 3 \text{ instead of } k = 2 \]

⇒ The study of QNMs may lead to a deeper understanding of black holes and quantum gravity.

Analytical derivation of asymptotic form of QNMs by Motl and Neitzke offered a new surprise

\[ \therefore \text{it heavily relied on the black hole singularity.} \]

It is intriguing that the unobservable region beyond the horizon influences the behavior of physical quantities.
GOAL

- Calculate first-order correction to asymptotic formula for QNMs.
  - by solving wave equation perturbatively for arbitrary spin of the wave.

We shall obtain agreement with results from

- numerical analysis for gravitational and scalar waves
  \[\text{[Nollert; Berti, Kokkotas]}\]

- a WKB analysis for gravitational waves.
  \[\text{[Maassen van den Brink]}\]
Metric:

\[ ds^2 = -h(r) \, dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2, \quad h(r) = 1 - \frac{2GM}{r} \]

Hawking temperature:

\[ T_H = \frac{1}{8\pi GM} = \frac{1}{4\pi r_0} \]

\( r_0 = 2GM \): radius of horizon.

A spin-\( j \) perturbation of frequency \( \omega \) is governed by the radial equation

\[ -h(r) \frac{d}{dr}\left(h(r) \frac{d\Psi}{dr}\right) + V(r) \Psi = \omega^2 \Psi \]

where \( V(r) \) is the “Regge-Wheeler” potential

\[ V(r) = h(r) \left( \frac{\hat{L}^2}{r^2} + \frac{(1 - j^2)r_0}{r^3} \right) \]

and \( \hat{L} = \ell(\ell + 1) \).

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<th>( j )</th>
<th>Wave</th>
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avoid integer values of $j$ throughout the discussion and only take the limit

$$j \rightarrow \text{integer}$$

at the end of the calculation.

“tortoise coordinate”

$$r_* = r + r_0 \ln \left( \frac{r}{r_0} - 1 \right)$$

Wave equation:

$$-\frac{d^2\Psi}{dr_*^2} + V(r(r_*))\Psi = \omega^2\Psi$$

to be solved along the entire real axis. At both ends the potential vanishes

$$V \rightarrow 0 \text{ as } r_* \rightarrow \pm\infty$$

∴ solutions behave as

$$\Psi \sim e^{\pm i\omega r_*}$$

For QNMs, demand

$$\Psi \sim e^{\mp i\omega r_*}, \quad r_* \rightarrow \pm\infty$$

assuming $\Re\omega > 0$. 
Let

\[ \Psi = e^{-i\omega r_*} f(r_*) \]

\[ \therefore \]

\[ f(r_*) \sim 1 \quad \text{as} \quad r_* \to +\infty \]

and near the horizon

\[ f(r_*) \sim e^{2i\omega r_*} \quad \text{as} \quad r_* \to -\infty \]

continue \( r \) analytically into the complex plane and define the boundary condition at the horizon in terms of the monodromy of \( f(r_*(r)) \) around the singular point \( r = r_0 \),

\[ \mathcal{M}(r_0) = e^{-4\pi\omega r_0} \]

along a contour running counterclockwise.

Deform contour in complex \( r \)-plane so that it either lies

- beyond the horizon (\( \Re r < r_0 \)) or
- at infinity (\( r \to \infty \)).

\[ \Rightarrow \] monodromy only gets a contribution from the segment lying beyond the horizon.
Change variables to
\[ z = \omega(r_\ast - i\pi r_0) = \omega(r + r_0 \ln(1 - r/r_0)) \]
(choose branch s.t. \( z \to 0 \) as \( r \to 0 \).)
The potential can be written as a series in \( \sqrt{z} \),
\[
V(z) = -\frac{\omega^2}{4z^2} \left( 1 - j^2 + \frac{3\ell(\ell + 1) + 1 - j^2}{3} \frac{2z}{\omega r_0} \right) + \ldots \]
(formal expansion in \( 1/\sqrt{\omega} \)).
Solve the wave equation perturbatively.
⇒ write wavefunction as a perturbation series in \( 1/\sqrt{\omega} \).
Zeroth order:
\[
\frac{d^2\Psi^{(0)}}{dz^2} + \left( \frac{1 - j^2}{4z^2} + 1 \right) \Psi^{(0)} = 0
\]
Solutions:
\[
f_{\pm}^{(0)}(z) = e^{iz} \Psi_{\pm}^{(0)} = e^{iz} \sqrt{\frac{\pi z}{2}} J_{\pm j/2}(z)
\]
Monodromy to zeroth order

\[ \mathcal{M}(r_0) = -\frac{\sin(3\pi j/2)}{\sin(\pi j/2)} = -(1 + 2 \cos(\pi j)) \]

⇒ discrete set of complex frequencies (QNMs)

\[ \frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln(1 + 2 \cos(\pi j)) + o(1/\sqrt{n}) \]

[Motl and Neitzke]
First order

expand wavefunction in $1/\sqrt{\omega}$

$$\psi = \psi^{(0)} + \frac{1}{\sqrt{-\omega r_0}} \psi^{(1)} + o(1/\omega)$$

First-order correction obeys

$$\frac{d^2\psi^{(1)}}{dz^2} + \left(\frac{1 - j^2}{4z^2} + 1\right) \psi^{(1)} = \sqrt{-\omega r_0} \delta V \psi^{(0)}$$

$$\delta V(z) = \frac{1 - j^2}{4z^2} + \frac{1}{\omega^2} V(r(z))$$

Solutions:

$$\psi^{(1)}_{\pm}(z) = C \psi^{(0)}_{\pm}(z) \int_0^z \psi^{(0)}_{\mp} \delta V \psi^{(0)}_{\pm}$$

$$-C \psi^{(0)}_{-}(z) \int_0^z \psi^{(0)}_{+} \delta V \psi^{(0)}_{-}$$

$$C = \frac{\sqrt{-\omega r_0}}{\sin(\pi j/2)}$$

integral along positive real axis on $z$-plane ($z > 0$).
monodromy to this order:

\[ \mathcal{M}(r_0) = -\frac{\sin(3\pi j/2)}{\sin(\pi j/2)} \left\{ 1 + \frac{i - 1}{2\sqrt{-\omega r_0}} e^{\pi j i/2} A \right\} \]

\[ \therefore \text{QNM frequencies} \]

\[ \frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln(1 + 2 \cos(\pi j)) \]

\[ + \frac{e^{\pi j i/2}}{\sqrt{n + 1/2}} A + o(1/n) \]

\[ A = (1 - i) \frac{3\ell(\ell + 1) + 1 - j^2}{24\sqrt{2}\pi^{3/2}} \frac{\sin(2\pi j)}{\sin(3\pi j/2)} \]

\[ \times \Gamma^2(1/4) \Gamma(1/4 + j/2) \Gamma(1/4 - j/2) \]

\[ \Rightarrow \text{well-defined finite limit as } j \to \text{integer.} \]
Scalar waves

\[ j \rightarrow 0^+ \]

\[
\frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln 3 \\
+ \frac{1 - i}{\sqrt{n + 1/2}} \frac{\ell(\ell + 1) + 1/3}{6\sqrt{2}\pi^{3/2}} \Gamma^4(1/4) + o(1/n)
\]

in agreement with numerical results
[Berti and Kokkotas]

Gravitational waves

\[ j \rightarrow 2 \]

\[
\frac{\omega_n}{T_H} = (2n + 1)\pi i + \ln 3 \\
+ \frac{1 - i}{\sqrt{n + 1/2}} \frac{\ell(\ell + 1) - 1}{18\sqrt{2}\pi^{3/2}} \Gamma^4(1/4) + o(1/n)
\]

in agreement with the results from

- a WKB analysis
  [Maassen van den Brink]

- numerical analysis
  [Nollert]
Kerr black holes
Extend above to rotating (Kerr) black holes

- NOT straightforward!

Bohr’s correspondence principle

\[ \delta M = \hbar \mathcal{R} \omega \]

and first law of black hole mechanics

\[ \delta M = T_H \delta S_{BH} + \Omega \delta J \]

⇒ asymptotic expression

[Hod]

\[ \mathcal{R} \omega = T_H \ln 3 + m \Omega \]

\( m \): azimuthal eigenvalue of wave
\( \Omega \): angular velocity of horizon.

Some numerical results ⇒

[Berti, Cardoso, Kokkotas, Onozawa]

\[ \mathcal{R} \omega = m \Omega \]

CONFLICT!
GOAL
Analytic solution to the wave (Teukolsky) equation

• valid for asymptotic modes bounded from above by $1/a$

$$a = \frac{J}{M}$$

$J$: angular momentum, $M$: mass of Kerr black hole.

Calculation valid for

$$a \ll 1$$

includes Schwarzschild case ($a = 0$).

Results

• confirm Hod’s expression

• do not necessarily contradict numerical results
  (may be valid in asymptotic regime $1/a \lesssim \omega$)

• In Schwarzschild limit ($a = 0$)
  – range of frequencies extends to infinity
  – our expression reduces to the expected form
Metric

\[ ds^2 = - \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 + \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2 \theta \left( r^2 + a^2 + \frac{2Ma^2r \sin^2 \theta}{\Sigma} \right) d\phi^2 \]

\[ \Sigma = r^2 + a^2 \cos^2 \theta \quad , \quad \Delta = r^2 - 2Mr + a^2 \]

\( M \): mass of black hole

Newton’s constant \( G = 1. \)

Roots of \( \Delta \)

\[ r_{\pm} = M \pm \sqrt{M^2 - a^2} \]

radius of horizon \( r_h = r_+ \).

Angular velocity

\[ \Omega = \frac{a}{2Mr_+} \]

Hawking temperature

\[ T_H = \frac{1 - r_-/r_+}{8\pi M} \]
Small perturbations are governed by the Teukolsky wave equation

\[
\left( \frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right) \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \phi} + \left( \frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right) \frac{\partial^2 \psi}{\partial \phi^2} - \frac{1}{\Delta s} \frac{\partial}{\partial r} \left( \Delta^{s+1} \frac{\partial \psi}{\partial r} \right) \\
- 2s \left( \frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \frac{\partial \psi}{\partial t} \\
- \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) - 2s \left( \frac{a(r - M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right) \frac{\partial \psi}{\partial \phi} \\
+(s^2 \cot^2 \theta - s)\psi = 0
\]

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Solution

\[\psi = e^{-i\omega t} e^{im\phi} S(\theta) f(r)\]
Angular equation:

\[
\frac{1}{\sin \theta} \left( \sin \theta \ S'' \right)'
\]

\[
+ \left( a^2 \omega^2 \cos^2 \theta - \frac{m^2}{\sin^2 \theta} - 2a\omega s \cos \theta - \frac{2ms \cos \theta}{\sin^2 \theta} - s^2 \cot^2 \theta \right) S = -(A + s)S
\]

\(A\): separation constant (eigenvalue)

Radial equation:

\[
\frac{1}{\Delta_s} \left( \Delta^{s+1} f' \right)' + V(r)f = (A + a^2 \omega^2)f
\]

where

\[
\Delta V(r) = (r^2 + a^2)^2 \omega^2 - 4aMr\omega m + a^2 m^2
\]

\[+ 2ia(r - M)ms - 2iM(r^2 - a^2)\omega s + 2ir\omega s \Delta\]

simplify by placing horizon at \(r = 1\)

\[2M = 1 + a^2\]

Roots of \(\Delta\)

\[r_- = a^2 \quad , \quad r_+ = 1\]
Solve the two wave equations by expanding in $a$

- keep terms up to $o(a)$
- assume $\omega$ is large but bounded from above by $1/a$,

$$1 \lesssim \omega \lesssim 1/a$$

$\omega$ is in an intermediate range (asymptotic in Schwarzschild limit $a \to 0$)

Solutions to angular equation to lowest order: spin-weighted spherical harmonics, and

$$A = \ell(\ell + 1) - s(s + 1) + o(a\omega)$$

To express the radial equation in terms of the tortoise coordinate, define

$$f(r) = \Delta_0^{-s/2} \frac{R(r)}{\sqrt{r(\omega r - am)}}$$

$\Delta_0 = r(r - 1)$ (NB: $\Delta = \Delta_0 + o(a^2)$).

$$r = \sqrt{-\frac{2z}{\omega}} + o(1/\omega)$$

$\Rightarrow$ radial equation to lowest order in $1/\sqrt{\omega}$ in
terms of $R$,

$$\frac{d^2 R}{dz^2} + \left\{ 1 + \frac{3is}{2z} + \frac{4 - s^2 - 4iams}{16z^2} \right\} R = 0$$

to be solved along the entire real axis.

Whittaker’s equation!

Monodromy around $r = 1$

$$\mathcal{M}(1) = e^{4\pi(\omega - ma)} = A$$

$$\Re \omega = \frac{1}{4\pi} \ln(1 + 2 \cos \pi s) + ma + o(a^2)$$

in agreement with Hod’s formula

- for gravitational waves ($s = -2$)
- in the small-$a$ limit

$$\Omega \approx a, \quad T_H \approx \frac{1}{4\pi}$$

NB: QNMs bounded from above by $1/a$. 
AdS Black Holes

AdS/CFT correspondence:
⇒ QNMs for AdS black hole expected to correspond to perturbations of the dual CFT. Establishment of correspondence hindered by difficulties in solving the wave equation.

- In 3d: Hypergeometric equation \(\therefore\) solvable
  
  \[\text{[Cardoso, Lemos; Birmingham, Sachs, Solodukhin]}\]

- In 5d: Heun equation \(\therefore\) unsolvable.

- Numerical results in 4d, 5d and 7d
  
  \[\text{[Horowitz, Hubeny; Starinets; Konoplya]}\]
Asymptotic form of QNMs of large AdS black holes

Approximation to the wave equation valid in the high frequency regime.

- In 3d: exact equation.
- In 5d: Heun equation $\rightarrow$ Hypergeometric equation, as in low frequency regime.
  - analytical expression for asymptotic form of QNM frequencies
  - in agreement with numerical results.
These expressions may also be easily obtained by considering the monodromies around the singularities of the wave equation.

- singularities lie in the unphysical region.
  - In three dimensions, they are located at the horizon \( r = r_h \), where \( r_h \) is the radius of the horizon, and at the black hole singularity, \( r = 0 \).
  - In higher dimensions, it is necessary to analytically continue \( r \) into the complex plane. The singularities lie on the circle \( |r| = r_h \).
  - similar to asymptotically flat space where analytic continuation of \( r \) yields asymptotic form of QNMs [Motl and Neitzke].

- It is curious that unphysical singularities determine the behavior of QNMs.
The wave equation (with $m = 0$) is

$$\frac{1}{R^2} \frac{1}{r} \partial_r \left( r^3 \left( 1 - \frac{r_h^2}{r^2} \right) \partial_r \Phi \right) - \frac{R^2}{r^2 - r_h^2} \partial_t^2 \Phi + \frac{1}{r^2} \partial_x^2 \Phi = 0$$

One normally solves this in physical interval:

$$r \in [r_h, \infty)$$

Instead, we shall solve it inside the horizon

$$0 \leq r \leq r_h$$

Solution:

$$\Phi = e^{i(\omega t - px)} \psi(y), \quad y = \frac{r^2}{r_h^2}$$

where $\psi$ satisfies

$$\left( y(1 - y) \frac{\psi'}{\psi} \right)' + \left( \frac{\hat{\omega}^2}{1 - y} + \frac{\hat{p}^2}{y} \right) \psi = 0$$

for $0 < y < 1$.

Two solutions obtained by examining behavior near the horizon ($y \to 1$),

$$\psi_\pm \sim (1 - y)^{\pm i\hat{\omega}}$$
A different set obtained by studying behavior at black hole singularity \((y \to 0)\)

\[ \Psi \sim y^{\pm i \hat{p}} \]

For QNMs, \(\Psi\) ingoing at the horizon

\[ \Psi \sim \Psi_{-} \text{ as } y \to 1 \]

By writing

\(\Psi(y) = y^{\pm i \hat{p}}(1 - y)^{-i \tilde{\omega}}F(y)\)

we deduce

\[ y(1 - y)F'' + \{1 \pm 2i \hat{p} - (2 - 2i(\tilde{\omega} \mp \hat{p})y\} F' \]

\[ + (\tilde{\omega} \mp \hat{p})(\tilde{\omega} \mp \hat{p} + i)F = 0 \]

Solution:

\[ F(y) = 2F_{1}(1 - i(\tilde{\omega} \mp \hat{p}), -i(\tilde{\omega} \mp \hat{p}); 1 \pm 2i \hat{p}; y) \]

- near the horizon \((y \to 1)\): mixture of ingoing and outgoing waves.
- blows up at infinity \((y \to \infty)\).
For a QNM, we demand that $F(y)$ be a Polynomial

⇒ takes care of both limits $y \to 1, \infty$

⇒

\[ \hat{\omega} = \pm \hat{p} - i n \quad , \quad n = 1, 2, \ldots \]

⇒

\[ F(y) = 2F_1(1 - n, -n; 1 \pm 2i\hat{p}; y) \]

is a Polynomial of order $n - 1$.

\[ \therefore \text{constant at } y = 1, \text{ as desired and } \]

\[ \therefore
\]

\[ F(y) \sim y^{n-1} \sim y^{i(\hat{\omega} \mp \hat{p})^{-1}} \text{ as } y \to \infty \]

so $\Psi \sim y^{-1}$ as $y \to \infty$, as expected.
A monodromy argument

Let $\mathcal{M}(y_0)$ be the monodromy around the singular point $y = y_0$ computed along a small circle centered at $y = y_0$ running counterclockwise. For $y = 1$,

$$\mathcal{M}(1) = e^{2\pi \hat{\omega}}$$

For $y = 0$,

$$\mathcal{M}(0) = e^{\mp 2\pi \hat{p}}$$

Since the function vanishes at infinity, the two contours around the two singular points $y = 0, 1$ may be deformed into each other without encountering any singularities,

$$\therefore \quad \mathcal{M}(1) \mathcal{M}(0) = 1$$

$$\therefore \quad e^{2\pi (\hat{\omega} \mp \hat{p})} = e^{2\pi in} \quad (n \in \mathbb{Z})$$

same QNM frequencies as before, if we demand $\Im \hat{\omega} < 0$. 
\textbf{AdS}_5

Wave equation with $m = 0$

\[
\frac{1}{r^3} \partial_r (r^5 \hat{f}(r) \partial_r \Phi) - \frac{R^4}{r^2 \hat{f}(r)} \partial_t^2 \Phi - \frac{R^2}{r^2} \vec{\nabla}^2 \Phi = 0
\]

\[
\hat{f}(r) = 1 - \frac{r_h}{r^4}
\]

Solution:

\[
\Phi = e^{i(\omega t - \vec{p} \cdot \vec{x})} \Psi(r)
\]

change coordinate $r$ to $y$,

\[
y = \frac{r^2}{r_h^2}
\]

Wave equation:

\[
(y^2 - 1) \left( y(y^2 - 1) \psi' \right)' + \left( \frac{\hat{\omega}^2}{4} y^2 - \frac{\hat{p}^2}{4} (y^2 - 1) \right) \psi = 0
\]

Two solutions by examining behavior near the horizon ($y \to 1$),

\[
\psi_\pm \sim (y - 1)^{\pm i\hat{\omega}/4}
\]
Different set by studying behavior at large $r$ ($y \to \infty$)

$$\Psi \sim y^{h_\pm}, \quad h_\pm = 0, -2$$

so one of the solutions contains logarithms. For QNMs, we are interested in analytic solution

$$\Psi \sim y^{-2} \text{ as } y \to \infty$$

By considering the other (unphysical) singularity at $y = -1$,

$\Rightarrow$ another set of solutions

$$\Psi \sim (y + 1)^{\pm \hat{\omega}/4} \text{ near } y = -1$$

Write wavefunction as

$$\Psi(y) = (y - 1)^{-i\hat{\omega}/4}(y + 1)^{\pm \hat{\omega}/4} F(y)$$

$\Rightarrow$ Two sets of modes with same $\Im \hat{\omega}$, but opposite $\Re \hat{\omega}$.

$F(y)$ satisfies the Heun equation

$$y(y^2 - 1)F'' + \left\{ \left( 3 - \frac{i \pm 1}{2} \hat{\omega} \right)y^2 - \frac{i \pm 1}{2} \hat{\omega}y - 1 \right\} F' + \left\{ \frac{\hat{\omega}}{2} \left( \pm \frac{i\hat{\omega}}{4} + 1 - i \right)y - (i \mp 1)\frac{\hat{\omega}}{4} - \frac{\hat{p}^2}{4} \right\} F = 0$$
Solve in a region in the complex $y$-plane containing $|y| \geq 1$
(includes physical regime $r > r_h$)

For large $\hat{\omega}$: constant terms in Polynomial coefficients of $F'$ and $F$ small compared with other terms

∴ they may be dropped.

∴ wave eq. may be approximated by Hypergeometric equation

$$(y^2 - 1)F'' + \left\{ \left(3 - \frac{i \pm 1}{2} \hat{\omega} \right)y - \frac{i \pm 1}{2} \hat{\omega} \right\} F'$$

$$+ \frac{\hat{\omega}}{2} \left( \pm \frac{i \hat{\omega}}{4} \mp 1 - i \right) F = 0$$

in asymptotic limit of large frequencies $\hat{\omega}$.

Analytic solution:

$$F_0(x) = 2F_1(a_+, a_-; c; (y + 1)/2)$$

$$a_{\pm} = 1 - \frac{i \pm 1}{4} \hat{\omega} \pm 1 \quad , \quad c = \frac{3}{2} \pm \frac{1}{2} \hat{\omega}$$
For proper behavior at $y \to \infty$, demand that $F$ be a *Polynomial*.

\[
 a_+ = -n \quad , \quad n = 1, 2, \ldots 
\]

\[
 F \sim y^n \sim y^{-a_+} 
\]

\[
 \psi \sim y^{-i\hat{\omega}/4} y^{\pm \hat{\omega}/4} y^{-a_+} \sim y^{-2} 
\]

as expected.

\::: QNM frequencies

\[
 \hat{\omega} = 2n(\pm 1 - i) 
\]

in agreement with numerical results.
**Monodromy argument**

If the function has no singularities other than $y = \pm 1$, the contour around $y = +1$ may be unobstructedly deformed into the contour around $y = -1$,

$$\mathcal{M}(1)\mathcal{M}(-1) = 1$$

Since

$$\mathcal{M}(1) = e^{\pi\hat{\omega}/2}, \quad \mathcal{M}(-1) = e^{\mp i\pi\hat{\omega}/2}$$

and using $\Im\hat{\omega} < 0$, we deduce

$$\hat{\omega} = 2n(\pm 1 - i)$$

same as before.

**Higher dimensions**

Wave equation possesses more than two singularities on the circle $|r| = r_h$ in the complex $r$-plane.

\[\therefore\text{ A simple monodromy argument does not appear to be applicable.}\]
Massive modes

In the limit $m \to \infty$, expect from numerical analysis [Starinets]

\[
\frac{\omega_n}{T_H} \sim 2\pi (\pm 1 - i)(n + h_+ - \frac{3}{2}) , \quad n = 1, 2, \ldots
\]

where $h_\pm = 1 \pm \sqrt{1 + m^2 R^2 / 4}$.

→ confirm this analytically
→ show corrections are $o(1/m) \sim o(1/h_+)$.

◊◊◊

Zeroth-order solutions:

$\mathcal{K}_\pm = (x + 1)^{-a_\pm} F(a_\pm, c - a_\mp; a_\pm - a_\mp + 1; 1/(x + 1))$

where

\[
a_\pm = h_\pm - \frac{1 + i}{4} \hat{\omega} , \quad c = \frac{3}{2} - \frac{i}{2} \hat{\omega} , \quad x = \frac{r^2 / r_h^2 - 1}{2}
\]

Choose $\mathcal{K}_+$, since it leads to $\Psi \to 0$ as $x \to \infty$. 
At horizon \((x \to 0)\),
\[
\mathcal{K}_+ \sim A_0 + B_0 x^{1-c}
\]
where
\[
A_0 = \frac{\Gamma(1 - c)\Gamma(1 - a_- + a_+)}{\Gamma(1 - a_-)\Gamma(1 - c + a_+)}
\]
\[
B_0 = \frac{\Gamma(c - 1)\Gamma(1 + a_+ - a_-)}{\Gamma(a_+)\Gamma(c - a_-)}
\]
Demand
\[
B_0 = 0
\]
\[
\tilde{\omega}_n = -2(1 + i)(n + h_+ - \frac{3}{2}) \quad n = 1, 2, \ldots
\]
agrees with numerical results \([\text{Starinets}]\)
First-order correction:

\[
F_1(x) = K_-(x) \int_x^\infty dx' \frac{K_+(x') \mathcal{H}_1 K_+(x')}{\mathcal{W}(x')}
\]

\[
- K_+(x) \int_x^\infty dx' \frac{K_-(x') \mathcal{H}_1 K_+(x')}{\mathcal{W}(x')}
\]

\[
\mathcal{H}_1 = \frac{1}{2x + 1} \left( \frac{1}{2} \frac{d}{dx} + (i - 1) \frac{\tilde{\omega}}{4} + \frac{\hat{p}^2}{4} \right)
\]

At horizon

\[
F_1(x) \sim A_1 + B_1 x^{1-c} \quad (x \to 0)
\]

where

\[
B_1 = \frac{\Gamma(c - 1)\Gamma(1 + a_+ - a_+)}{\Gamma(a_-)\Gamma(c - a_+)} \int_0^\infty dx \frac{K_+(x)\mathcal{H}_1 F_0(x)}{\mathcal{W}(x)}
\]

quasi-normal frequencies are solutions of

\[
B_0 + B_1 = 0
\]
Results for $\hat{p} = 0$.

$n = 1$

$$\hat{\omega}_1 = -2(1 + i) \left( h_+ - \frac{1}{2} + \frac{i}{4h_+} \right)$$

$n = 2$

$$\hat{\omega}_2 = -2(1 + i) \left\{ h_+ + \frac{1}{2} + \frac{12i - 1}{32h_+} + \ldots \right\}$$

$n \geq 3$

$\mathcal{K}_+ = F_0$ contains a polynomial of order $n - 1$

$\hookrightarrow$ complexity of the calculation increases with increasing $n$.

$\hookrightarrow$ One obtains terms of order $h_+^{n-1}$.

$\hookrightarrow$ cancel order-by-order, leaving an expression of order $1/h_+$

$$\hat{\omega}_n = -2(1 + i)(n + h_+ - \frac{3}{2}) + o(1/h_+)$$

$\hookrightarrow$ calculation is uninspiring.

Desirable to have an expression for the first-order correction manifestly of order $1/h_+$.
Conclusions

• QNMs have long been known **numerically**

• Recently started to understand QNMs **analytically**

• Behavior appears to rely on
  – unobservable, or even
  – unphysical
  singularities

• Bohr’s correspondence principle applies to
  Schwarzschild black holes
  – generalizes to slowly rotating Kerr black holes.

Future

• Fully understand behavior of QNMs in general space-time backgrounds.

• What do QNMs tell us about the AdS/CFT correspondence?

• If not Bohr’s correspondence principle, then what?
  – Quantum Gravity: The Holy Grail!